

A

SOLUTION

OF

KEPLER'S PROBLEM;

BY

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MDCCLVI.

ERRATA.

Page 23d, from the bottom of the page:

line 4. *read* 0.3190764

line 3. *read* 2'.084857

line 1. *read* $50^{\circ} 51'.093797$, that is,
 $50^{\circ} 51' 5''.62782$.

Page 24th. from the bottom;

line 2. *read* $50^{\circ} 51'.093797$.



A
S O L U T I O N
O F
KEPLER'S PROBLEM.

THO' the problem proposed by the celebrated *Kepler* has been solved by several Mathematicians of great note: yet, as this is of considerable use in astronomy, it is hoped the following solution may be agreeable to some; as it requires a less degree of knowledge in the more difficult parts of Mathematics, than the other methods do.

P R O P. I. *Fig. 1.*

Let there be a streight line AB, and CD a portion of a curve wholly concave towards AB, and draw AC, BD parallel to each other, meeting the curve in C, D; let CE, a tangent

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gent to the curve at C , meet BD in E ; join AD , AE , BC , and let AE meet the curve in F : the triangle ABC will be greater than the sector ACF , but less than the sector ACD .

JOIN CD . Because AC , BE , are parallel, the triangles, ABC , AEC , will be equal: but the triangle AEC is greater than the sector ACF ; therefore the triangle ABC is greater than the sector ACF . Again, because AC , BD , are parallel, the triangles, ABC , ADC , are equal: but the triangle ADC is less than the sector ACD ; therefore the triangle ABC is less than the sector ACD .

P R O P. II. *Fig. 2.*

Let there be a curve AEB , wholly concave towards the straight line AB , and let C , D , be two points in the line AB ; draw DE to any point E

E in the curve, and draw CF parallel to DE, meeting the curve in F; let EG, a tangent to the curve at E, meet CF in G, and join DF, DG, and let DG meet the curve in H; let the point D be between the points, A, C: the sector ACE will be greater than the sector ADH, but less than the sector ADF; and the sector BCE will be greater than the sector BDF, but less than the sector BDH.

BECAUSE the triangle DEC is greater than the sector EDH, let the sector ADE be added to both; and the space ACE will be greater than the sector ADH: and, because the triangle DEC is less than the sector EDF, let the sector ADE be added to both; and the sector ACE will be less than the sector ADF.

AGAIN, because the sector ACE together with the sector BCE, is equal to the sector ADF together with the sector BDF, and the
sector

sector ACE is less than the sector ADF ; therefore the sector BCE is greater than the sector BDF : and, because the sector ACE together with the sector BCE, is equal to the sector ADH together with the sector BDH, and the sector ACE is greater than the sector ADH ; therefore the sector BCE will be less than the sector BDH.

PROP. III. PROBL. I. *Fig. 3.*

Let there be a given circle AEB, and let D be a given point within the circle; and from D let there be drawn DE, DF, to two given points, E, F, in the circle: granting the quadrature of the circle, it is required to draw a line DG meeting the arc FE in G; so that the sector EDG may be to the sector GDF in a given ratio, suppose that of m to n .

LET C be the center of the circle, and join CE, CF; draw DH, DK, perpendicular to CE,

CE, CF, meeting CE, CF in H, K ; and let EL, FM, be tangents to the circle at the points, E, F.

BECAUSE the points, D, E, F, are given, and the circle is given, the sector DEF will be given ; and, because the sector EDG is to the sector GDF in the given ratio of m to n , the sectors, EDG, GDF, will, each of them, be given in magnitude. In the tangent EL take the point L towards F ; so that, joining HL, the triangle HEL may be equal to the sector EDG : again, in the tangent FM take the point M towards E ; so that, joining KM, the triangle KFM will be equal to the sector FDG : it is evident, the points, L, M, will be given. Join DL, DM, meeting the circle in N, O ; the point G will fall between the points, N, O. For, because EL, FM, are tangents to the circle at E, F, the angles HEL, KFM, will be right ; and therefore DH, EL, will be parallel : likewise DK, FM, will be parallel ; therefore the triangles, DEL, HEL, will be equal, and likewise the triangles, DFM, KFM, will be equal : and therefore the triangle DEL will be equal to the sector DEG, and the triangle DFM equal to

to

to the sector DFG; therefore the sector DEG will be greater than the sector DEN, and the sector DFG greater than the sector DFO; therefore the line DG will fall between the lines, DN, DO; therefore the point G will fall between the points, N, O.

AGAIN, Because the points, L, M are given, and the point D likewise given, the sectors, DEN, DFO, will be given; and, because the sectors, DEG, DFG, are given, the sectors, DNG, DOG, will be given; and therefore the sector NDG will be to the sector GDO in a given ratio, suppose that of p to q . The problem, therefore, will be reduced to this again, To divide the sector NDO by the line DG, so that the sector NDG may be to the sector GDO in the given ratio of p to q . And thus, repeating the operation, the point G will be found within a very narrow limit.

N. B. THE limit ON might have been found by taking in the tangents, EL, FM, the point L towards F, and the point M towards E; so that, joining DL, DM, meeting the circle in N, O, the triangle DEL would be equal to the sector EDG, and the triangle DFM

DFM equal to the sector DFG: but the other way points out the method of calculation.

PROP. IV. PROBL. 2. *Fig. 4.*

Let there be a semicircle whose diameter is AB and center C, and let D be a point in the diameter: granting the quadrature of the circle, it is required to draw a line DE meeting the circle in E, so that the semicircle may be to the sector BDE in a given ratio, suppose that of p to q .

SUPPOSE the problem solved. Let the semicircle be to the sector BCF as p is to q ; join DF; draw CG parallel to DF, meeting FG, a tangent to the circle at F, in G, and join DG meeting the circle in H; let CG meet the circle in K, and join DK, FK, CH.

BECAUSE the semicircle is to the sector BDE as p to q , that is, as the semicircle to the
sector

sector BCF, the sector BDE will be equal to the sector BCF: but the sector BCF is [*Prop. 2.*] greater than the sector BDK, but less than the sector BDH; therefore the sector BDE is greater than the sector BDK, and less than the sector BDH; therefore the line DE falls between the lines, DH, DK; and therefore the point E falls between the points, H, K. Again, because the sector BDE is equal to the sector BCF, and the sector BCK is common to both, the space KCDEK will be equal to the sector KCF; and, because the triangles, KCD, KCF, are equal, (because KC, DF, are parallel), therefore the sector KDE is equal to the space KEFK; and, because the triangles, GDK, GFK, are equal, taking the common space GHK from both, the sector KDH will be equal to the space KEFK together with the space GFHG; but the sector KDE is equal to the space KEFK, therefore the sector EDH is equal to the space GFHG.

BECAUSE the semicircle is to the sector BCF as p to q , therefore the angle BCF is given, and therefore the angle FCD is likewise given: and, in the triangle FCD, because the sides, CF, CD, are given, and likewise

ways the angle FCD, the angles CDF, CFD will be given; and therefore the angles BCK, KCF will be given; therefore the arcs BK, KF are given: because FG is a tangent to the circle at the point F, the angle CFG will be a right angle: and because the angle FCG is given, and likewise the side CF given, the sides CG, CF will be given: in the triangle GCD, because the sides GC, CD are given, and the angle GCD given, the angles CGD, CDG will be given: and in the triangle GCH, because the sides GC, CH and the angle CGH is given, the angle GCH will be given; and therefore the arc HK will be given: and because the arc FK is given, therefore the arc FH is given. Again, because the arc FK is given, the sector FCK will be given, and likewise the triangle FCK given, therefore the space KEFK will be given; and because the triangles, GFC, GHC, are given, the space GFCHG will be given; and because the arc FH is given, the sector FCH will be given therefore the space GFHG will be given.

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From

From this, the following construction may be deduced.

CONSTRUCTION.

In the semicircle take the point F, and let the arc AFB be to the arc BF as p is to q ; join CF, DF, and draw CG parallel to DF, meeting FG, a tangent to the circle, at F in G; and the circle in K; join DG meeting the circle in H, and join FK, DK; draw the line DE [*Prop. 3.*] meeting the circle in E, so that the sector KDE may be to the sector EDH as the space KEFK to the space GFHG: the semicircle will be to the sector BDE as p is to q .

BECAUSE the sector KDE is to the sector EDH, as the space KEFK to the space GFHG; the sector EDH will be to the sector KDH, as the space KEFK to the sum of the spaces KEFK, GFHG: but, because
the

the triangles, GDK, GFK are equal, and the space GHK common to both, the sector KDH will be equal to the sum of the spaces KEFK, GFHG ; therefore the sector KDE is equal to the space KEFK. And, because the triangle KDC is equal to the triangle KFC, the space KCDE will be equal to the sector KCF; therefore (adding the sector BCK to both) the sector BDE will be equal to the sector BCF. Again, because the arc AFB is to the arc BF as p is to q , the semicircle will be to the sector BCF as p is to q ; therefore the semicircle will be to the sector BDE as p is to q . Q. E. D.

BUT, as this would require a good deal of trigonometrical calculation, the following method may be used; which will give the point sought very nearly, when this problem is of use in the planetary system.

PROP. V. PROBL. 3. *Fig. 5.*

Let there be a semicircle whose diameter is AB and centre C, and let D be a point in the diameter not
very

very excentric ; granting the quadrature of the circle, it is required to draw a line, DE, meeting the circle in E, so that the semicircle may be to the sector BDE in a given ratio, suppose that of p to q ,

SUPPOSE the problem solved ; and let the semicircle be to the sector BCF as p is to q ; join DF, and draw CG parallel to DF meeting FG a tangent to the circle at F in G ; join DG, and let CG, DG meet the circle in H, K : join DH, FH ; draw CM parallel to DH meeting HM a tangent to the circle at H in M ; draw DL perpendicular to CH meeting CH in L ; join LE, and let FN perpendicular to CH meet CH in N.

BECAUSE the semicircle is to the sector BDE as p is to q , that is, as the semicircle to the sector BCF, the sector BDE will be equal to the sector BCF : and because the sector BCH is common to both, the space HCDE will be equal to the sector HCF ; but because CH, DF are parallel, the triangle DCH is equal to the triangle FCH ; therefore

fore the sector EDH will be equal to the space contained by the arc HF and the chord FH. Again, because the sector BDE is equal to the sector BCF, and the sector BCF is [*Prop. 2.*] greater than the sector BDH, and less than the sector BDK, the sector BDE will be greater than the sector BDH, and less than the sector BDK: therefore the point E is in the arc HK between the points H, K. [Let the arc HK be called the limiting arc]. Because the point D is not very excentric, the limiting arc will be little. In the orbit of *Mercury*, the most excentric of all the planets, the limiting arc will be less than sixteen minutes in that part of the orbit where it is greatest; therefore EH may be considered as coinciding with the tangent to the circle at the point H; and therefore the triangle ELH will be equal to the sector EDH, that is, equal to the space contained by the arc FH and the chord FH; therefore the rectangle LHE will be double of the space contained by the arc FH and the chord FH. Because the rectangle contained by CH and the arc FH is double of the sector CFH, and the rectangle contained by CH, FN is double of the triangle CFH; the
rectangle

rectangle contained by CH and the excess of the arc FH above FN will be double of the space contained by the arc FH and the chord FH; therefore the rectangle LHE will be equal to the rectangle contained by CH and the excess of the arc FH above FN; therefore LH will be to HC as the excess of the arc FH above FN to HE: but because the triangles DLH, CHM are similar, LH will be to HC as DL or FN to HM; therefore FN will be to HM as the excess of the arc FH above FN to HE.

From this, the following construction may be deduced.

CONSTRUCTION.

Let the semicircle be to the sector BCF as p is to q ; join DF, draw CH parallel to DF meeting the circle in H; join DH, draw CM parallel to DH meeting HM a tangent to the circle at H in M; and let FN perpendicular to CH meet CH in N; in the tangent HM

HM take HE towards F, so that FN may be to HM as the excess of the arc FH above FN to HE; join DE: the semicircle will be to the sector BDE as p is to q .

LET DL, perpendicular to CH, meet CH in L; join FH, LE. Let CH meet FG, a tangent to the circle at F, in G; join DG meeting the circle in K.

BECAUSE the triangles DLH, CHM are similar, LH will be to HC as DL or FN to HM; that is, as the excess of the arc FH above FN to HE: therefore the rectangle LHE will be equal to the rectangle contained by HC and the excess of the arc HF above FN: but, because the rectangle contained by CH and the arc HF is double of the sector HCF, and the rectangle contained by CH, FN is double of the triangle CFH; therefore the rectangle contained by CH and the excess of the arc FH above FN will be double of the space contained by the arc FH and the chord FH; therefore the rectangle LHE will be double of the space contained by the arc FH and the chord FH;
and

and therefore the triangle LHE will be equal to the space contained by the arc FH and the chord FH: but, because DL, HE are parallel, the triangles DHE, LHE are equal; therefore the triangle DHE is equal to the space contained by the arc FH and the chord FH: and, because DF, CH are parallel, the triangles CDH, CFH are equal; therefore the space HCDE will be equal to the sector CFH. Let the sector ECH be added to both, and the sector BDE will be equal to the sector BCF: but, because the sector BCF is greater than the sector BDH and less than the sector BDK, therefore the sector BDE is greater than the sector BDH and less than the sector BDK, and therefore DE falls between DH, DK: and, because the point D is not very excentric, the limiting arc HK will be little; therefore the tangent HE may be considered as coinciding with the arc HK: because the sector BDE is equal to the sector BCF, the semicircle will be to the sector BDE as the semicircle to the sector BCF: but the semicircle is to the sector BCF as p is to q ; therefore the semicircle is to the sector BDE as p is to q .
Q. E. D.

Kepler

Kepler first of all discovered that the planets revolved in ellipses round the sun placed in one of the *foci*, and that they described equal areas in equal times round the sun. Let the semi-ellipse, [Fig. 6.] whose greater *axis* is AP, *focus* S, and centre C, represent half the orbit of a planet round the sun in S; and suppose the planet at the point K in its orbit; join SK : half the periodic time of the planet round the sun, is to the time the planet moves from A to P, as the area of the semi-ellipse to the area ASK; and therefore to find the place of the planet at any given time, it is necessary to find the position of the right line SK, which shall cut off the area ASK proportional to the time, that is, To draw the line SK so that the area of the semi-ellipse may be to the area ASK, as half the periodic time of the planet round the sun to the given time.

FROM K let fall KH perpendicular to AP, meeting the semicircle described upon AP in G, and join SG : it is evident, from the nature of the ellipse and circle, that the semicircle is to the sector ASG as the semi-ellipse to the sector ASK; therefore the semicircle is to the sector ASG as half the periodic time of

the planet round the sun to the time the planet moves from A to K: the problem therefore is reduced to this, To draw the line SG meeting the semicircle in G, so that the semicircle may be to the sector ASG as half the periodic time of the planet round the sun to the given time. In the semicircle, take the arc AB, so that the arc ABP may be to the arc AB as half the periodic time of the planet round the sun to the given time. Join CB; the semicircle therefore will be to the sector ACB as half the periodic time of the planet round the sun to the given time, that is, as the semicircle to the sector ASG; therefore the sectors ACB, ASG, are equal; join CG.

The angle ACB is called by *Kepler* the mean anomaly, the angle ACG the anomaly of the excentric, and the angle ASK the co-equate or true anomaly. The problem therefore is reduced to this: The mean anomaly of a planet being given, to find the anomaly of the excentric and the coequate anomaly.

PROP.

PROP. VI. PROBL. IV. *Fig. 6.*

Let AP be the greater axis of a planet's orbit, S the focus the place of the sun, A the aphelion, P the perihelion; upon AP let the semicircle ABP be described; let C be the center, and let the angle ACB be the mean anomaly of the planet at any given time: it is required to find the anomaly of the excentric.

JOIN SB, and draw CD parallel to SB meeting the circle in D: join SD; in BD, take the arc DG, so that the sine of the angle BCD may be to the tangent of the angle CDS, as the excess of the arc ED above its sine to the arc DG; and join CG: the angle ACG will be nearly the anomaly of the excentric.

JOIN SG; draw BE perpendicular to CD meeting CD in E, and draw CF parallel to SD meeting DF a tangent to the circle at D in F.

BECAUSE

BECAUSE BE is the sine of the angle BCD, and DF the tangent of the angle DCF, that is, of the angle SDC; BE will be to DF as the excess of the arc BD above BE to the arc DG; therefore [*Prop. 5.*] the sector ASG is equal to the sector ACB; therefore the angle ACG is the anomaly of the excentric.

THE computation is as follows: In the triangle BCS, as the sum of the sides BC, CS, is to the difference of the sides BC, CS, so is the tangent of half the angle ACB to the tangent of half the difference of the angles, CSB, CBS; therefore the angles, CSB, CBS will be given, that is, the angles, ACD, DCB, will be given. Again, in the triangle CSD, the sum of the sides, CD, CS, is to the difference of the sides CD, CS, as the tangent of half the angle ACD to the tangent of half the difference of the angles, CSD, CDS; therefore the angle CDS will be given. Again, because as the sine of the angle BCD is to the tangent of the angle CDS, so is the excess of the arc BD above its sine to the arc GD; say as the radius is to the sine of the angle BCD, so is $57^{\circ}.2957795$ &c. the number of degrees in an angle subtended by

by an arc equal to the radius, to the number of degrees in an angle subtended by an arc equal to the sine of the angle BCD; [let this angle be called A]. Again, as the sine of the angle BCD to the tangent of the angle CDS, so is the excess of the angle BCD above the angle A, to the angle GCD. Therefore the angle ACG, the anomaly of the excentric, will be given.

E X A M P L E I.

IN the orbit of *Mercury*, the mean distance is to the excentricity as 100000 to 20589. Suppose the mean anomaly from the aphelion to be 60° , it is required to find the anomaly of the excentric. In the triangle BCS, as 120589, the sum of BC, CS, is to 79411 the difference of the sides BC, CS, so is the tangent of 30° , half the sum of the angles, CSB, CBS, to the tangent of half the difference of the angles CSB, CBS.

The log. tang. of 30° is	9.7614394	
The log. of 79411 is	4.8998807	
The sum is	14.6613201	
The log. of 120589 is	5.0813077	
The difference	9.5800124	is the log.

log. tang. of $20^{\circ} 49'.00894$ half the difference of the angles CSB, CBS ; therefore the angle CSB or ACD is $50^{\circ} 49'.00894$, and the angle CBS or BCD is $9^{\circ} 10'.99106$. Again, in the triangle DCS, as 120589 the sum of DC, CS, is to 79411 the difference of DC, CS, so is the tangent of $25^{\circ} 24'.50447$ half the sum of the angles CSD, CDS, to the tangent of half the difference of the angles CSD, CDS.

The log. tang. of $25^{\circ} 24'.50447$ is	9.6767070
The log. of 79411 is	4.8998807
The sum is	14.5765877
The log. of 120589 is	5.0813077
The difference	9.4952800

is the log. tang. of $17^{\circ} 22'.21093$ half the difference of the angles CSD, CDS ; therefore the angle CDS is $8^{\circ} 2'.29354$. Again, as the radius is to the sine of $9^{\circ} 10'.99106$, so is $57^{\circ} .2957795$ &c. the number of degrees in an angle subtended by an arc equal to the radius, to the number of degrees and minutes in an angle subtended by an arc equal to the sine of $9^{\circ} 10'.99106$.

The

The log. of 57.2957795 is 1.7581226

The log. fine of $9^{\circ} 10'.99106$ is 9.2030097

The sum is 10.9611323

The log. of rad. is 10.0000000

The difference 0.9611323

is the log. of $9^{\circ} 8'.6349999$, the number of degrees and minutes contained in an angle subtended by an arc equal to the fine of $9^{\circ} 10'.99106$. The angle A therefore is $9^{\circ} 8'.6349999$; the excess therefore of the angle BCD above the angle A is $2'.35606$. Again, as the fine of $9^{\circ} 10'.99106$ is to the tangent of $8^{\circ} 2'.29354$, so is the angle $2'.35606$ to the angle DCG.

The log. of $2'.35606$ is 0.3721863

The log. tang. of $8^{\circ} 2'.29354$ is 9.1498998

The sum is 9.5220861

The log. fine of $9^{\circ} 10'.99106$ is 9.2030097

The difference 0.3190764

is the log. of $2'.084057$ the angle DCG, the angle ACG the anomaly of the excentric is $50^{\circ} 51'.093797$, that is $50^{\circ} 51' 5''.6$

LET

7.62702

LET SM, perpendicular to CG, meet CG in M: It is evident, because the sector ACG is common to both the sectors ASG, ACB, that, if the sector ASG be equal to the sector ACB, the triangle SCG will be equal to the sector BCG; and therefore the line SM will be equal to the arc BG: if the sector ASG be greater than the sector ACB, the triangle SCG will be greater than the sector BCG; and therefore the line SM will be greater than the arc BG: and, if the sector ASG be less than the sector ACB, the triangle SCG will be less than the sector BCG; and therefore the line SM will be less than the arc BG: that is, if the arc AG be greater, equal, or less, than the true anomaly of the excentric, the line SM will be greater, equal, or less, than the arc BG; and therefore, the less the difference is between the line SM and the arc BG, the less will the difference be between the arc AG and the anomaly of the excentric.

BECAUSE the triangles GCH, CSM, are similar, CG is to GH as CS to SM; that is, the radius is to the sine of $50^{\circ} 51'.093893$ as 20589 to CM.

The

The log. of 20589 is 4.3136353

The log. sine of $50^{\circ} 51'.093797$ is 9.8895890

The sum is 14.2032243

The log. of rad. is 10.0000000

The difference 4.2032243

is the log. of SM. Again, as CA to SM, so is $57^{\circ}.2957795, \&c.$ the number of degrees in an angle subtended by an arc equal to CA, to the number of degrees in an angle subtended by an arc equal to SM.

The log. of $57^{\circ}.2957795, \&c.$ is 1.7581226

The log. of SM is 4.2032243

The sum is 5.9613469

The log. of CA 5.0000000

The difference 0.9613469

is the log. of $9^{\circ} 8'.90625$ the number of degrees and minutes in an angle subtended by an arc equal to SM: but, because the arc AB is 60° , and the arc AG is $50^{\circ} 51'.093797$, the arc BG will be $9^{\circ} 8'.906203$; the difference therefore between SM and the arc BG

D

is

is 0'.000047 ; the difference therefore is less than the 354th part of a second.

Mr. *Machin*, in his solution of this problem in the Philosophical Transactions, Number 447. makes the anomaly of the excentric to be $129^{\circ}.14846$, when the mean anomaly reckoned from the perihelion is 120° ; and therefore, if the mean anomaly reckoned from the aphelion be 60° , the anomaly of the excentric will be $50^{\circ}.85154$, that is $50^{\circ} 51'.0924$.

In order to determine the difference between SH and the arc BG according to this computation,

The log. of 20589 is	4.3136353
The log. sine of $50^{\circ} 51'.0924$ is	9.8895888
The sum is	14.2032241
The log. of rad. is	10.0000000
The difference	4.2032241

is the log. of SM. Again, as CA to SM, so is $57^{\circ}.2957795$, &c. the number of degrees in an angle subtended by an arc equal to CA, to the number of degrees in an angle subtended by an arc equal to SM.

The

The log. of $57^{\circ}.2957795$, &c. is 1.7581226

The log. of SM is 4.2032241

The sum is 5.9613467

The log. of CA is 5.0000000

The difference 0.9613467

is the log. of $9^{\circ} 8'.905998$, the number of degrees and minutes in an angle subtended by an arc equal to SM: but, because the arc AB is 60° , and the arc AG is $50^{\circ} 51'.0924$, the arc BG will be $9^{\circ} 8'.9076$; the difference therefore between SM and the arc BG is $0'.001602$, very nearly one tenth of a second, and is more than thirty four times greater than the former difference.

BECAUSE AS is equal to the sum of BC, CS, and PS equal to the difference of BC, CS; it is evident, that, if, from the log. tang. of half the angle ACB, the difference of the logarithms of AS, SP be subtracted, the remainder will be the log. tang. of an angle, which, if added to half the angle ACB, will give the angle ASB or ACD; and, if subtracted from half the angle ACB, will give the angle CBS or BCD.

EX-

E X A M P L E II.

IN the orbit of *Mars*, the mean distance is to the excentricity as 152369 to 14100. Supposing the mean anomaly to be 1° , it is required to find the anomaly of the excentric.

The log. tang. of $30'$, half the angle ACB, is 7.9408584

The difference of the logs. of AS, SP is 0.0806086

The difference 7.8602498

is the log. tang. of $24'.9196814$, half the difference of the angles CSB, CBS; therefore, the angle CSB, that is the angle ACD, is $54'.9196814$; and the angle CBS, that is the angle BCD, is 5.0803186 . Again, to determine the angle CDS,

The log. tang. of half the angle ACD is 7.8958249

The difference of the logs. of AS, SP is 0.0806086

The difference 7.8152163

is the log. tang. of $22'.4641426$, half the difference of the angles CSD, CDS; therefore

fore the angle CDS, that is the angle DCF, is 4'.0956981.

AGAIN, as the radius is to the fine of 5'.0803186, so is $57^{\circ} 29' 7795$, &c. the number of degrees in an angle subtended by an arc equal to the radius, to the number of degrees and minutes in an angle subtended by an arc equal to the fine of 5'.0803186.

The log. of $57^{\circ} 29' 7795$ is 1.7581226

The log. fine of 5'.0803186 is 7.1633313

The sum is 8.9214539

The log. of radius is 10.0000000

The difference —2.9214539

is the log. of 5'.0073162, the number of minutes contained in the angle A; the excess therefore of the angle BCD, above the angle A, is 0'.0930024. Again, as the fine of the angle BCD is to the tangent of the angle DCF, so is 0'.0930024 to the angle DCG.

The log. of 0'.0930024 is —2.9684941

The log. tang. of the angle DCF is 7.1622795

The sum is 6.1307736

The log. fine of the angle BCD is 7.1633313

The difference 2.9674423

is

is the log. of 0.0927774 ; the angle DCG therefore is 0.0927774 ; therefore the angle ACG, the anomaly of the excentric, is 55.0124588 .

E X A M P L E III,

SUPPOSING the mean anomaly in the same orbit to be 45° , it is required to find the anomaly of the excentric.

The log. tang. of $22^\circ 30'$ is 9.6172243

The diff. of the logs. of AS, SP is 0.0806085

The difference 9.5366158

is the log. tang. of $18^\circ 59'.1327325$, half the difference of the angles CSB, CBS; the angle CSB, that is the angle ACD, therefore is $41^\circ 29'.1327325$; and the angle CBS or BCD is $3^\circ 30'.8672675$.

The log. tang. of half the angle

ACD is 9.5783203

The diff. of the logs. of AS, SP is 0.0806085

The difference 9.4977118

is the log. tang. of $17^\circ 27'.7082672$, half the difference of the angles CSD, CDS; the angle CDS or DCF, therefore, is $3^\circ 16'.858099$.

$3^{\circ} 16'.858099$. Again, as radius is to the fine of the angle BCD, so is $57^{\circ}.2957795$, &c. to the angle A.

The log. of $57^{\circ}.2957795$, &c. is 1.7581226

The log. fine of the angle BCD is 8.7874623

The sum is 10.5455849

The log. of rad. is 10.0000000

The difference 0.5455849

is the log. of $3^{\circ}.5122459$; the angle A, therefore, is $3^{\circ} 30'.734754$; the excess of the angle BCD above the angle A is $0'.1325135$. Again, as the fine of the angle BCD is to the tangent of the angle SDC or DCF, so is $0'.1325135$ to the angle DCG.

The log. of $0'.1325135$ is -1.1222601

The log. tang. of the angle SDC is 8.7583537

The sum is 7.8806138

The log. fine of the angle BCD is 8.7874623

The difference -1.0931515

is the log. of $0'.1239228$; the angle DCG, therefore, is $0'.1239228$; therefore the angle ACG, the anomaly of the excentric, is $41^{\circ} 29'.2566553$.

EX-

E X A M P L E IV.

AGAIN, in the same orbit supposing the mean anomaly to be 100° degrees, it is required to find the anomaly of the excentric.

The log. tang. of 50° half the angle ACB is 10.0761865

The difference of the logs. of AS, SP is 0.0806086

The difference 9.9955779

is the log. tang. of $44^\circ 42'.4982192$, half the difference of the angles CSB, CBS; therefore the angle CSB, that is the angle ACD, is $94^\circ 42'.4982192$; and the angle CBS, that is the angle BCD, is $5^\circ 17'.5017808$. Again, to determine the angle CDS,

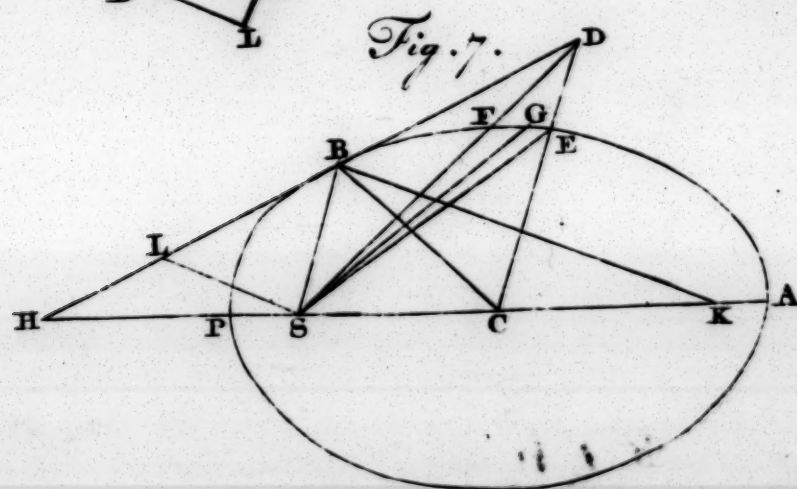
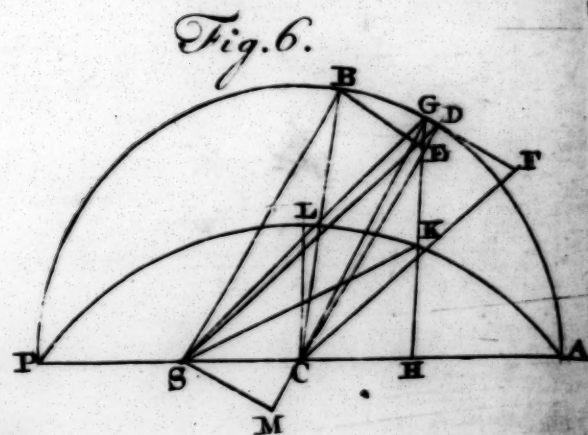
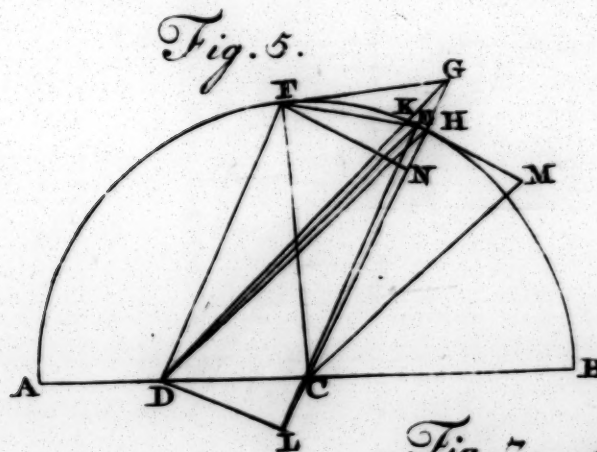
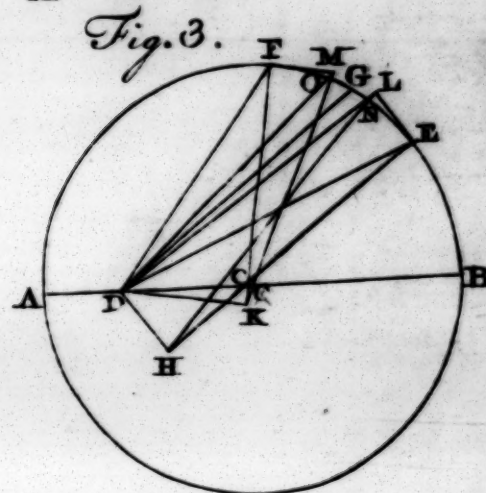
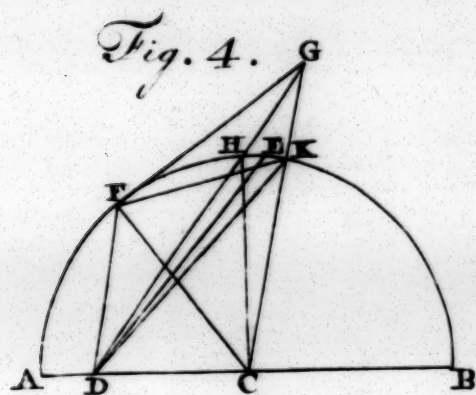
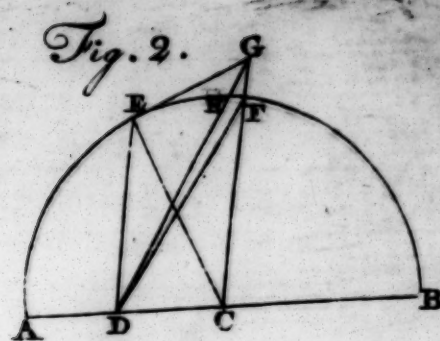
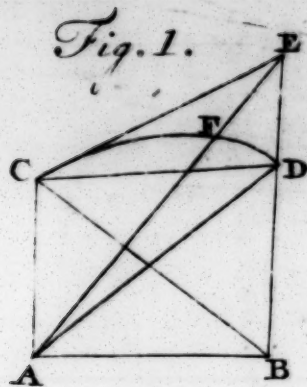
The log. tang. of half the angle ACD is 10.0557285

The difference of the logs. of AS, SP is 0.0806086

The difference 9.9751199

is the log. tang. of $43^\circ 21'.5819834$, half the difference of the angles CSD, CDS; therefore the angle CDS, that is the angle DCF, is $3^\circ 59'.6671212$.

AGAIN,



6: 18: 36
18

30315.

978
659
325
978

AGAIN, as the radius is to the sine
of $5^{\circ} 17'.5017808$ the angle BCD, so is
 $57^{\circ}.2997795$ to the angle A.

The log. of $57^{\circ}.2957795$ is 1.7581226

The log. sine of $5^{\circ} 17'.5017808$ is 8.9642381

The sum is 10.7223607

The log. of radius is 10.0000000

The difference 0.7223607

is the log. of $5^{\circ} 16'.600752$ the number of
degrees and minutes contained in the angle
A; the excess therefore of the angle BCD
above the angle A is $0'.9010288$. Again,
as the sine of the angle BCD is to the tangent
of the angle DCF, so is $0'.9010288$ to the
angle DCG.

The log. of $0'.9010288$ is —1.9547387

The log. tang. of the angle DCF is 8.8440431

The sum is 8.7987818

The log. sine of the angle BCD is 8.9642381

The difference —1.8345437

is the log. of $0'.6831934$; the angle DCG
therefore is $0'.6831934$; therefore the
angle ACG the anomaly of the excentric is
 $94^{\circ} 43'.1814126$.

E

THE

THE three last examples are taken from Dr. *Keil's* astronomical Lectures, Lecture 23d. and the numbers agree very nearly with his.

IN the orbits of *Mercury* and *Mars*, if the excess of the angle BCD above the angle A be added to the angle ACD, the sum will be nearly the anomaly of the excentric reckoned from the aphelion.

IN orbits of small excentricity, the angle ACD is nearly the anomaly of the excentric; therefore the following rule will give the anomaly of the excentric very nearly.

X FROM the logarithmic tangent of half the mean anomaly, subtract the difference of the logarithms of the aphelion and perihelion distances; the remainder is the logarithmic tangent of an angle, which call B: to the angle B, add half the mean anomaly; the sum will give very nearly the anomaly of the excentric.

E X A M P L E V.

IN the earth's orbit, the mean distance is to the excentricity as 100000 to 1691. Suppose the mean anomaly from the aphelion to be 30° , it is required to find the anomaly of the excentric.

The

The log. tang. of 15° is 9.4280525

The diff. of the logs. of AS, SP is 0.0146892

The difference 9.4133633

is the log. tang. of $14^\circ 31'.3670421$, half the difference of the angles CSB, CBS; therefore the angle CSB, that is the angle ACD, is $29^\circ 31'.367042$; therefore the anomaly of the excentric is nearly $29^\circ 31'.3670421$, that is $29^\circ 31'.22''.022526$, which agrees very nearly with Dr. *Keil's* numbers in the forefaid Lecture.

EXAMPLE VI.

AGAIN, in the earth's orbit, suppose the mean anomaly to be 60° , it is required to find the anomaly of the excentric.

The log. tang. of 30° is 9.7614394

The diff. of the logs. of AS, SP is 0.0146892

The difference 9.7467502

is the log. tang. of $29^\circ 10'.081873$, half the difference of the angles CSB, CBS; therefore the angle CSB, that is the angle ACD, is $59^\circ 10'.081873$; therefore the anomaly of the excentric is nearly $59^\circ 10'.081873$.

EX-

E X A M P L E VII.

In the orbit of *Venus*, the mean distance is to the excentricity as 10000000 to 69855. Suppose the mean anomaly reckoned from the aphelion to be 60° , it is required to find the anomaly of the excentric.

The log. tang. of 30° is 9.7614394

The diff. of the logs. of AS, SP is 0.0060677

The difference 9.7553717

is the log. tang. of $29^\circ 39'.2739959$, half the difference of the angles CSC, CBS; therefore the angle CSB, that is the angle ACD, is $59^\circ 39'.2739959$; therefore the anomaly of the excentric is nearly $59^\circ 39'.2739959$, that is $59^\circ 39'.16''.439754$.

If the mean anomaly reckoned from the perihelion be 120° , the anomaly of the excentric would be nearly $120^\circ 20'.43''.560246$, which agrees very nearly with Mr. *Machin's* numbers in the forecited transaction.

THE anomaly of the excentric being found, the coequate or true anomaly will be found by the resolution of the triangle GCS: Thus, from the log. tang. of half the anomaly of the excentric, subtract the difference of the
logs,

logs. of the aphelion and perihelion distances, the remainder will be the log. tang. of an angle; to this angle add half the anomaly of the excentric; let the sum be called the angle C; to the log. tang. of the angle C, add half the sum of the logs. of the aphelion and perihelion distances; from this sum, subtract the log. of the mean distance; the remainder will be the log. tang. of the coequate or true anomaly.

LET CL be the lesser axis of the planet's orbit. Because, from the nature of the ellipse, the square of CL is equal to the rectangle ASP, the log. of CL will be equal to half the sum of the logs. of AS, SP: and, because the tangent of the angle KSH is to the tangent of the angle GSH as HK to HG; that is, from the nature of the ellipse, as LC to CA; therefore, if to the log. tang. of the angle GSH, the log. of CL be added, and from the sum the log. of AC be subtracted, the remainder will be the log. tang. of the angle KSH.

AGAIN, the sine of the true anomaly is to the sine of the anomaly of the excentric, as the lesser axis of the orbit to the distance of the planet from the sun.

BECAUSE

BECAUSE CL is to HK as CG to GH, that is, as the radius to the sine of the anomaly of the excentric, and HK is to KS as the sine of the true anomaly to the radius; therefore CL is to KS as the sine of the true anomaly to the anomaly of the excentric.

THE place of a planet in an elliptic orbit [granting the quadrature of the ellipse] may be found at any given time within a small limit, by the following theorem.

THEOREM. FIG. 7.

Let the ellipse, whose greater axis is AP, foci S, K, and center C, represent the orbit of a planet round the sun at S; and, supposing the periodic time of the planet round the sun to be known, and likewise the time the planet passed thro' the aphelion A: As the periodic time of the planet round the sun, is to the time elapsed since the planet passed thro' the point

point A, so let the area of the ellipse be to the sector ACB; join SB, and draw CD parallel to SB on the same side of AP that SB is; and let CD be equal to CA; join SD; let CD, SD, meet the ellipse in E, F: the true place of the planet is between the points E, F; that is, the planet is passed the point E, but not come to the point F.

LET G be the place of the planet; join SG, and join BD meeting AP in H, and join KB; draw SL parallel to KB meeting BD in L. Because CD is parallel to SB, CD will be to SB as CH to HS; therefore, twice CD will be to SB as twice CH to HS. But, because twice CD is equal to AP, that is, equal to KB together with BS, and twice CH is equal to KH together with HS; therefore KB together with BS will be to BS as KH together with HS is to HS; and therefore KB will be to BS as KH to HS, that is, as KB to SL; therefore BS, SL are equal: therefore the angle SBL is equal to the angle SLB,

SLB, that is, equal to the angle KBD; and therefore, from a known property of the ellipse, BD is a tangent to the ellipse at the point B; and therefore [*Prop.* 2.] the sector ACB is greater than the sector ASE, and less than the sector ASF: but, because G is the place of the planet, the area of the ellipse will be to the sector ASG as the periodic time of the planet round the sun is to the time elapsed since the planet passed thro' the point A, that is, as the area of the ellipse to the sector ACB; therefore the sector ASG is equal to the sector ACB; and therefore the sector ASG is greater than the sector ASE, and less than the sector ASF; therefore the line SG falls between the lines SE, SF; and therefore G, the place of the planet, is between the points E, F; therefore the planet has passed the point E, but not come to the point F.

F I N I S.

